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Constructing Maximal Cohen–Macaulay Modules from Cyclic Deficiency Modules

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Which structural conditions ensure the existence of a maximal Cohen–Macaulay module?

MAIN THEOREM (XIE)

Let $(A, \mathfrak{m})$ be a d-dimensional Noetherian local ring admitting a dualizing complex. Then

$$\boxed{\operatorname{depth} A = d-1 \quad \text{and} \quad K^{d-1}(A) \text{ cyclic}} \quad \Longrightarrow \quad \boxed{A \text{ admits an MCM module.}}$$

An MCM module is a nonzero finitely generated A-module M with $\operatorname{depth}_A M = d$.
(Notation: $K^i(M) = H^{-i}(\operatorname{Hom}_A(M, D_A))$ for a normalized dualizing complex D_A. Equivalently, if A is a homomorphic image of a Gorenstein local ring R of dimension n, then $K^i(M) := \operatorname{Ext}_R^{n-i}(M, R)$. $\omega_A := K^d(A)$ is the canonical module.)

01 Construct the module

Put $C = K^{d-1}(A)$. Let $X := D_A[d-1]$ be the dualizing complex shifted so that

$$H^{-1}(X) = \omega_A, \qquad H^0(X) = C, \qquad H^i(X) = 0, \text{ for } i \neq \{0, -1\}.$$

From the identification,

$$\operatorname{Hom}_{D(A)}(A, X) \cong H^0(X) = C$$

a generator $c \in C$ corresponds to a morphism $f\colon A \to X$.

$$A \xrightarrow{f} X \longrightarrow \operatorname{Cone}(f) \longrightarrow A[-1]$$

Since $H^0(f)\colon A \twoheadrightarrow C$ is surjective, the associated long exact sequence in cohomology gives

$$\operatorname{Cone}(f) \simeq M[-1], \qquad M := H^{-1}(\operatorname{Cone}(f)).$$

02 Why is M maximal CM?

Use the duality functor with appropriate shift $\mathbb{D}(-) := \mathbf{R}\operatorname{Hom}_A(-, X) = \mathbf{R}\operatorname{Hom}_A(-, D_A)[d-1]$. Biduality gives

$$\mathbb{D}(A) = X, \qquad \mathbb{D}(X) \simeq A.$$

Define the transpose of f as

$$f^\dagger\colon A \simeq \mathbb{D}(X) \xrightarrow{\mathbb{D}(f)} \mathbb{D}(A) = X.$$

Transposition $f \mapsto f^\dagger$ induces an A-linear automorphism of $\operatorname{Hom}_{D(A)}(A, X) \cong C$. It therefore sends a generator of C to another generator. Hence $f^\dagger$ is induced by another generator of C, and $H^0(f^\dagger)\colon A \twoheadrightarrow C$.

THE KEY STEP

The same surjectivity argument works for $f^\dagger$ and gives

$$\operatorname{Cone}(f^\dagger) \simeq N[-1], \qquad N := H^{-1}\big(\operatorname{Cone}(f^\dagger)\big).$$

Dualizing the cone triangle yields

$$\mathbb{D}(M) \simeq \operatorname{Cone}(f^\dagger),$$

so $\mathbb{D}(M)$ also has cohomology only in degree -1.

Local duality now gives

$$H^i_{\mathfrak{m}}(M) = 0 \quad (i < d), \qquad \boxed{\operatorname{depth}_A M = d.}$$

Notes beyond cyclicity.
If $\mu_A(C) = r > 1$, replacing $A \to C$ by $A^r \twoheadrightarrow C$ gives $\mu_A(K^{d-1}(M)) = r^2 - 1$. So increasing the number of generators amplifies the obstruction rather than resolving it. The cyclic case explains a special self-duality phenomenon.

03 A concrete kernel description

Suppose $x \in \operatorname{Ann}_A(C)$ is a nonzerodivisor on A, and put $B = A/xA$. There is an exact sequence

$$0 \longrightarrow \omega_A/x\omega_A \xrightarrow{\bar{\tau}} \omega_B \xrightarrow{\pi} C \longrightarrow 0.$$

Using a suitable lift $\tilde{\alpha} : A \to \omega_B$ of a surjection $\alpha : A \to C$ along π, we define

$$\Phi := (\bar{\tau}, \tilde{\alpha})\colon \omega_A \oplus A \twoheadrightarrow \omega_B$$

and we identified the derived construction with the kernel

$$\boxed{M \cong \ker(\Phi).}$$

WHEN A IS QUASI-GORENSTEIN

If $\omega_A \cong A$, then for *every* A-regular $x \in \operatorname{Ann}_A(C)$, ω_B has two minimal generators, and M is identified with the first syzygy of $\omega_B = \omega_{A/xA}$:

$$\boxed{M \cong \Omega^1_A(\omega_{A/xA}).}$$

which is MCM.

This recovers and extends the construction of Tavanfar–Shimomoto [1] for quasi-Gorenstein deformations of a three dimensional quasi-Gorenstein ring A with $K^2(A) = k$.

04 Extending Previous Constructions

Tavanfar–Shimomoto [2] proved the following theorem.

> Let $(A, \mathfrak{m})$ be a quasi-Gorenstein local ring which is a homomorphic image of a Gorenstein local ring. Suppose that the following condition holds:
>
> There exists a regular sequence $\mathbf{y}$ such that $A/\mathbf{y}A$ is a 3-dimensional quasi-Gorenstein local ring satisfying $H^2_{\mathfrak{m}}(A/\mathbf{y}A) = k$. (*)
>
> Then there exists some $x \in \operatorname{ann}_A K^{d-1}(A)$, such that $x, \mathbf{y}$ form a regular sequence of A and the first syzygy $\Omega^1\omega_{A/xA}$ of the canonical module $\omega_{A/xA}$ of A/xA is a maximal Cohen–Macaulay A-module of rank 2.

The original proof reduces to a 3-dimensional quotient $A/\mathbf{y}A$ and lifts the MCM module $\Omega^1_A(\omega_{A/(x,\mathbf{y})A})$ over $A/\mathbf{y}A$. The condition (*) encodes the deformation conditions ensuring that the relevant deficiency modules and syzygy lift to A, and the choice of $x \in \operatorname{Ann}_A K^{d-1}(A)$ depends on the sequence $\mathbf{y}$. Simpler proofs of these steps can be found in Xie [3].
Note that the condition (*) can be equivalently replaced by $\operatorname{depth} A = d-1$ and $e_{\mathfrak{m}}(K^{d-1}(A)) = 1$. Since $K^{d-1}(A)$ is CM, $e_{\mathfrak{m}}(K^{d-1}(A)) = 1$ implies cyclicity. Our theorem thus applies to give that $\Omega^1\omega_{A/xA}$ is MCM for *any* nonzerodivisor $x \in \operatorname{Ann}_A K^{d-1}(A)$, independently of the deformation sequence.

05 Examples from section rings

For an ample line bundle L on a projective variety X, write

$$R(X, L) = \bigoplus_{n \geq 0} H^0(X, L^n), \qquad A = R(X, L)_{R_+}.$$

CHARACTERISTIC ZERO

Let Z be a projective Calabi–Yau variety of dimension $r \geq 1$. In $Y = Z \times \mathbf{P}^2$, take a smooth divisor

$$X \in |cH + bF|, \qquad c \gg 0, \quad b > 3,$$

where H comes from an ample line bundle on Z and F from $\mathcal{O}_{\mathbf{P}^2}(1)$. For $L = \omega_X^{\otimes a}$, $a \geq 2$,

$$\dim A = r + 2, \qquad \operatorname{depth} A = r + 1, \qquad K^{r+1}(A) \cong k.$$

The main theorem supplies an MCM module.

POSITIVE CHARACTERISTIC

Over a perfect field, let Z be an elliptic curve or a K3 surface of finite height. For an ample line bundle $\mathcal{L}$ on Z, set

$$X = Z \times \mathbf{P}^1, \qquad L = \mathcal{L} \boxtimes \mathcal{O}_{\mathbf{P}^1}(1).$$

With $r = \dim Z$, again $\dim A = r + 2$, $\operatorname{depth} A = r + 1$, and $K^{r+1}(A) \cong k$. **The completion $\widehat{A}$ admits an MCM module by the main theorem, but no module-finite Cohen–Macaulay algebra** (Bhatt [1]).

SELECTED REFERENCES

[1] B. Bhatt, On the non-existence of small Cohen–Macaulay algebras, *J. Algebra* **411** (2014), 1–11.
[2] K. Shimomoto and E. Tavanfar, Remarks on the small Cohen–Macaulay conjecture and new instances of maximal Cohen–Macaulay modules, *J. Algebra* **634** (2023), 667–697.
[3] L. Xie, On an instance of the small Cohen–Macaulay conjecture, *J. Pure Appl. Algebra* **228** (2024), no. 3, 107491.
[4] L. Xie, *On an instance of the small Cohen–Macaulay conjecture II*, *Bull. London Math. Soc.* (2026), to appear.